Week 14

Sections 8.3, 8.4

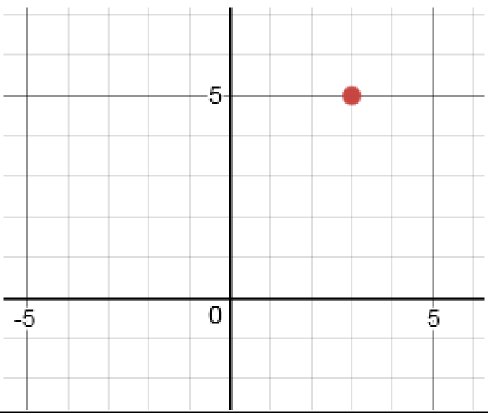
10, 16, 18, 34, 50, 66 (p. 610-611)
4, 6, 32, 34 (p. 617)

Review Exercises

Example

Graph the complex number $3 + 5i$ and find its modulus.

Solution



$$|z| = \sqrt{a^2 + b^2}$$

$$|z| = \sqrt{3^2 + 5^2}$$

$$|z| = \sqrt{9 + 25}$$

$$|z| = \sqrt{34}$$

Example

Given $z = -2 + i\sqrt{2}$. Sketch the complex numbers $z, 2z, -z$, and $\frac{1}{2}z$ on the same complex plane.

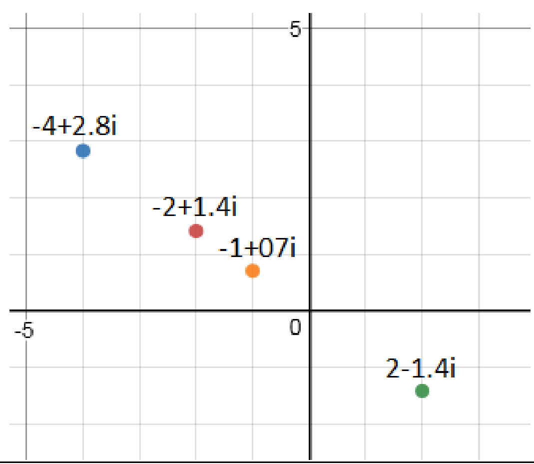
Solution

$$z = -2 + i\sqrt{2}$$

$$2z = 2(-2 + i\sqrt{2}) = -4 + 2i\sqrt{2} \approx -4 + 2.8i$$

$$-z = -(-2 + i\sqrt{2}) = 2 - i\sqrt{2} \approx 2 - 1.4i$$

$$\frac{1}{2}z = \frac{1}{2}(-2 + i\sqrt{2}) = -1 + \frac{\sqrt{2}}{2}i \approx -1 + 0.7i$$



<u>Example</u>

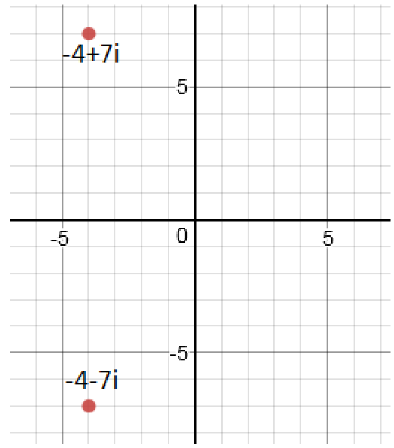
Given $z = -4 + 7i$. Sketch the complex number z and its complex conjugate z on the same complex plane.

<u>Solution</u>

The conjugate of $z = a + bi$ is $\bar{z} = a - bi$.

$z = -4 + 7i.$

$\bar{z} = -4 - 7i$



<u>Example</u>

Write the complex number $-3 + 3\sqrt{3}i$ in polar form with argument θ between 0 and 2π.

<u>Solution</u>

The number is located in quadrant II.

Find r.

$r = \sqrt{a^2 + b^2}$

$r = \sqrt{(-3)^2 + \left(3\sqrt{3}\right)^2}$

$r = \sqrt{9 + 27}$

$r = \sqrt{36}$

$r = 6$

Find θ.

$\tan\theta = \dfrac{b}{a}$

$\tan\theta = \dfrac{3\sqrt{3}}{-3}$

$\tan\theta = -\sqrt{3}$

In quadrant II,

$\theta = \dfrac{2\pi}{3}$

Use the formula: $a + bi = r(\cos\theta + i\sin\theta)$.

$$-3 + 3\sqrt{3}i = 6\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Example

Find the product $z_1 z_2$ and the quotient $\frac{z_1}{z_2}$. Express your answer in polar form.

$$z_1 = \sqrt{2}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$
$$z_2 = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

Solution

To multiply, use formula:

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

$$z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{5\pi}{3} + \frac{3\pi}{2}\right) + i\sin\left(\frac{5\pi}{3} + \frac{3\pi}{2}\right)\right)$$

$$z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{5\pi \cdot 2}{3 \cdot 2} + \frac{3\pi \cdot 3}{2 \cdot 3}\right) + i\sin\left(\frac{5\pi \cdot 2}{3 \cdot 2} + \frac{3\pi \cdot 3}{2 \cdot 3}\right)\right)$$

$$z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{10\pi}{6} + \frac{9\pi}{6}\right) + i\sin\left(\frac{10\pi}{6} + \frac{9\pi}{6}\right)\right)$$

$$\boldsymbol{z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{19\pi}{6}\right) + i\sin\left(\frac{19\pi}{6}\right)\right)}$$

Or, if we subtract 2π:

$$\boldsymbol{z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)}$$

To divide, use formula:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\left(\cos\left(\frac{5\pi}{3} - \frac{3\pi}{2}\right) + i\sin\left(\frac{5\pi}{3} - \frac{3\pi}{2}\right)\right)$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\left(\cos\left(\frac{5\pi \cdot 2}{3 \cdot 2} - \frac{3\pi \cdot 3}{2 \cdot 3}\right) + i\sin\left(\frac{5\pi \cdot 2}{3 \cdot 2} - \frac{3\pi \cdot 3}{2 \cdot 3}\right)\right)$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\left(\cos\left(\frac{10\pi}{6} - \frac{9\pi}{6}\right) + i\sin\left(\frac{10\pi}{6} - \frac{9\pi}{6}\right)\right)$$

$$\boldsymbol{\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)}$$

Example

Find the indicated power using DeMoivre's Theorem.

$$(1+i)^7$$

Solution

First, we have to write $1+i$ in $r(\cos\theta + i\sin\theta)$ form.

The number is located in quadrant I.

Find r.

$r = \sqrt{a^2+b^2}$

$r = \sqrt{1^2+1^2}$

$r = \sqrt{2}$

Find θ.

$\tan\theta = \frac{b}{a}$

$\tan\theta = \frac{1}{1}$

$\tan\theta = 1$

In quadrant I,

$\theta = \frac{\pi}{4}$

Use the formula: $a + bi = r(\cos\theta + i\sin\theta)$.

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Now use the DeMoivre's Theorem:

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$(1+i)^7 = \left(\sqrt{2}\right)^7\left(\cos 7\cdot\frac{\pi}{4} + i\sin 7\cdot\frac{\pi}{4}\right)$$

$$(1+i)^7 = 8\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$$(1+i)^7 = 8\sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$(1+i)^7 = 8 - 8i$$

Example

A pair of parametric equations are given:

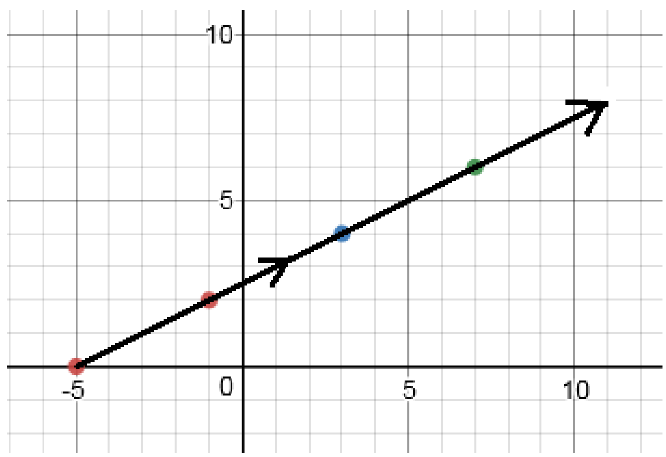
$$x = 4t - 5, \quad y = 2t$$
$$t \geq 0$$

a. Sketch the curve represented by the parametric equations. Use arrows to indicate the direction of the curve as t increases.
b. Find a rectangular-coordinate equation for the curve by eliminating the parameter.

Solution

a.

t	$x = 4t - 5$	$y = 2t$	(x, y)
0	$x = 4 \cdot 0 - 5 = -5$	$y = 2 \cdot 0 = 0$	$(-5, 0)$
1	$x = 4 \cdot 1 - 5 = -1$	$y = 2 \cdot 1 = 2$	$(-1, 2)$
2	$x = 4 \cdot 2 - 5 = 3$	$y = 2 \cdot 2 = 4$	$(3, 4)$
3	$x = 4 \cdot 3 - 5 = 7$	$y = 2 \cdot 3 = 6$	$(7, 6)$



b.

Solve $y = 2t$ for t.	Replace t in $x = 4t - 5$.
$\frac{y}{2} = \frac{2t}{2}$ $t = \frac{y}{2}$	$x = 4 \cdot \frac{y}{2} - 5$ $x = 2y - 5$ $x - 2y + 5 = 0$

Example

A pair of parametric equations are given:

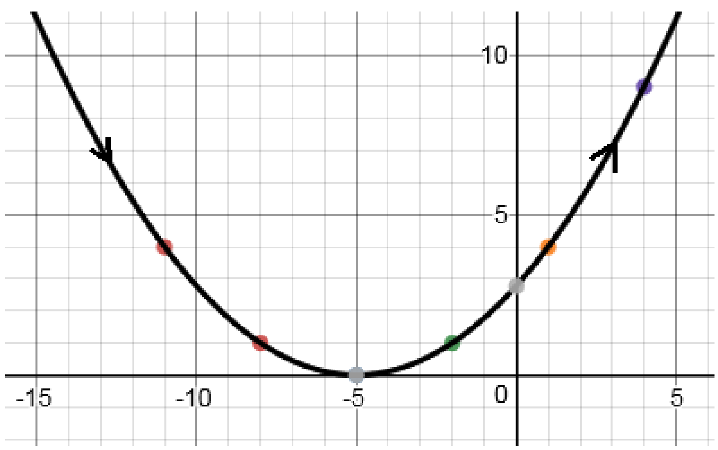
$$x = 3t - 2, \quad y = (t+1)^2$$

a. Sketch the curve represented by the parametric equations. Use arrows to indicate the direction of the curve as t increases.
b. Find a rectangular-coordinate equation for the curve by eliminating the parameter.

Solution

a.

t	$x = 3t - 2$	$y = (t+1)^2$	(x, y)
-3	$x = 3 \cdot (-3) - 2 = -11$	$y = (-3+1)^2 = 4$	$(-11, 4)$
-2	$x = 3 \cdot (-2) - 2 = -8$	$y = (-2+1)^2 = 1$	$(-8, 1)$
-1	$x = 3 \cdot (-1) - 2 = -5$	$y = (-1+1)^2 = 0$	$(-5, 0)$
0	$x = 3 \cdot 0 - 2 = -2$	$y = (0+1)^2 = 1$	$(-2, 1)$
1	$x = 3 \cdot 1 - 2 = 1$	$y = (1+1)^2 = 4$	$(1, 4)$
2	$x = 3 \cdot 2 - 2 = 4$	$y = (2+1)^2 = 9$	$(4, 9)$
3	$x = 3 \cdot 3 - 2 = 7$	$y = (3+1)^2 = 16$	$(7, 16)$



b.

Solve $x = 3t - 2$ for t.	Replace t in $y = (t+1)^2$.
$x + 2 = 3t$ $\dfrac{x+2}{3} = \dfrac{3t}{3}$ $t = \dfrac{x+2}{3}$	$y = \left(\dfrac{x+2}{3} + 1\right)^2$ $y = \left(\dfrac{x}{3} + \dfrac{2}{3} + 1\right)^2$ $y = \left(\dfrac{x}{3} + \dfrac{5}{3}\right)^2$

<u>Example</u>

Find parametric equations for the curve with the given properties.
The line with the slope -3, passing through $(-5, -7)$.

<u>Solution</u>

Infinitely many pairs of parametric equations can represent the same plane curve.

Method 1

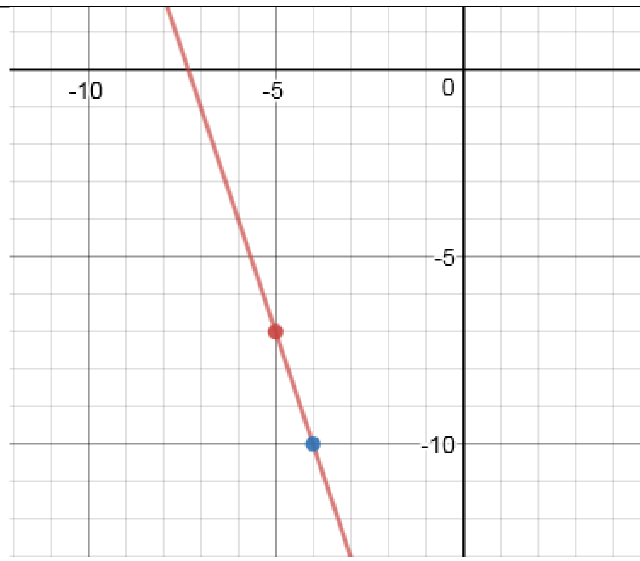
Given: $m = -3$, $(-5, -7)$

Find the equation of the line using the formula: $y - y_1 = m(x - x_1)$.

$$y - (-7) = -3\big(x - (-5)\big)$$
$$y + 7 = -3(x + 5)$$
$$y + 7 = -3x - 15$$
$$y = -3x - 22$$

$$\boldsymbol{x = t}$$
$$\boldsymbol{y = -3t - 22}$$

Method 2



From the point $(-5, -7)$, when x increases by 1, y decreases by -3. So,

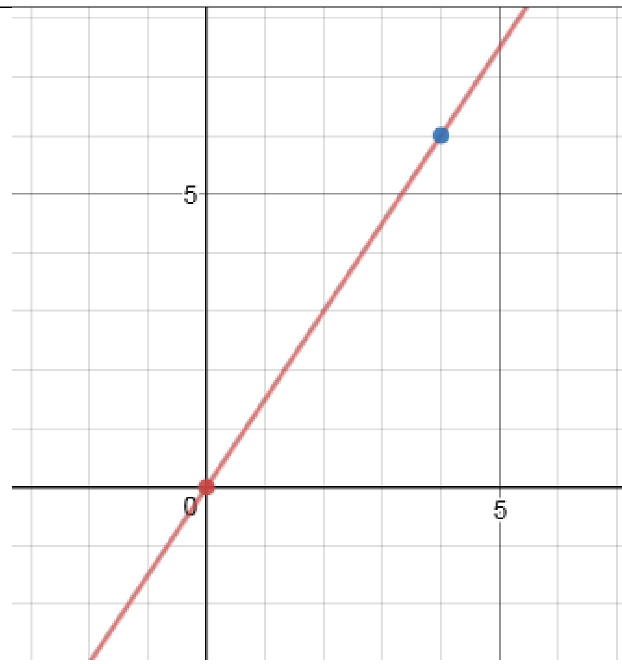
$$\boldsymbol{x = t - 5}$$
$$\boldsymbol{y = -3t - 7}$$

<u>Example</u>

Find parametric equations for the curve with the given properties.
The line passing through $(4, 6\ )$ and the origin.

<u>Solution</u>

Infinitely many pairs of parametric equations can represent the same plane curve.

<u>Method 1</u>	<u>Method 2</u>
Given: $(0, 0)\quad and \quad (4, 6)$ Find the slope: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6}{4} = \frac{3}{2}$ Find the equation of the line using the formula: $y - y_1 = m(x - x_1)$. $y - 0 = \frac{3}{2}(x - 0)$ $y = \frac{3}{2}x$ $\boldsymbol{x = t}$ $\boldsymbol{y = \frac{3}{2}t}$	 From the point $(0, 0)$, when x increases by 2, y increases by 3. So, $\boldsymbol{x = 2t}$ $\boldsymbol{y = 3t}$