Inverse Functions

Definition of an Inverse Function

The functions f and f^{-1} (read "f-inverse") are inverses of each other if $f\left(f^{-1}(x)\right) = x$ for every x in the domain of f^{-1} and $f^{-1}\left(f(x)\right) = x$ for every x in the domain of f.

To find the inverse of a function, we need to interchange x and y.

$$f(x) = y \quad \Leftrightarrow \quad f^{-1}(y) = x$$

An Example that Verifies that a Function is an Inverse of the Other

Show that the function $f(x) = 3x + 5$ is the inverse of the function $g(x) = \frac{x-5}{3}$.

Solution

According to the definition, two functions are inverses of each other if $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$.

We need to show that the composition of both f and g, and g and f equals x.

$$f\left(g(x)\right) = 3\left(\boldsymbol{\frac{x-5}{3}}\right) + 5 = x - 5 + 5 = x$$

$$g\left(f(x)\right) = \frac{\boldsymbol{3x+5} - 5}{3} = \frac{3x}{3} = x$$

So, these two functions are inverses of each other.

How to Find the Inverse of a Function

Example 1

Find the inverse of the function $f(x) = 3x + 1$

Step 1

Replace $f(x)$ with y.

$$f(x) = 3x + 1$$
$$y = 3x + 1$$

Step 2

Interchange x and y.

$$x = 3y + 1$$

Step 3

Solve for y.

$$x = 3y + 1$$
$$-1 \qquad -1$$

$$x - 1 = 3y$$

$$\frac{x-1}{3} = \frac{3y}{3}$$

$$\frac{x}{3} - \frac{1}{3} = y$$

Step 4

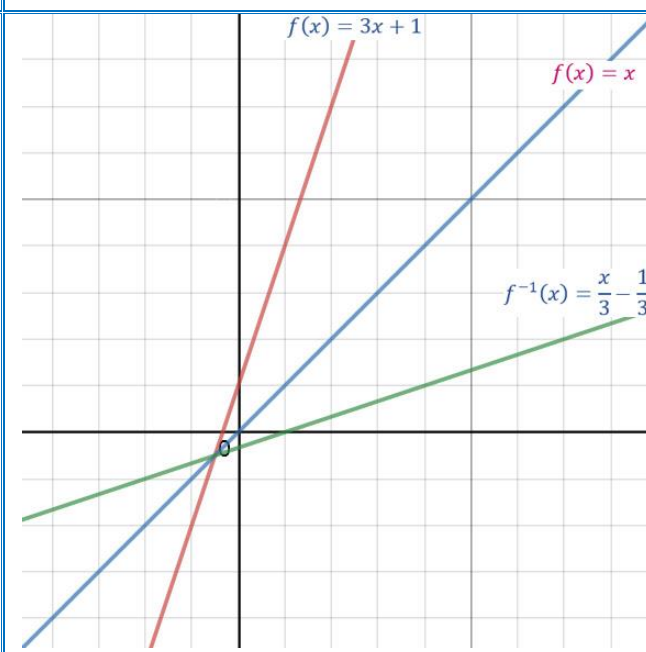
Replace y with $f^{-1}(x)$.

$$f^{-1}(x) = \frac{x}{3} - \frac{1}{3}$$

So, the inverse of $f(x) = 3x + 1$ is

$$f^{-1}(x) = \frac{x}{3} - \frac{1}{3}$$

The graph of an inverse function is symmetric to the given function with respect to the line $f(x) = x$.



How to Find the Inverse of a Function

Example 2

Find the inverse of the function $f(x) = \frac{2x+5}{x-4}$

Step 1

Replace $f(x)$ with y.

$$f(x) = \frac{2x+5}{x-4}$$

$$y = \frac{2x+5}{x-4}$$

Step 2

Interchange x and y.

$$x = \frac{2y+5}{y-4}$$

Step 3

Solve for y.

$$(y-4)(x) = \left(\frac{2y+5}{y-4}\right)(y-4)$$

$$xy - 4x = 2y + 5$$
$$xy - 2y = 4x + 5$$
$$y(x-2) = 4x + 5$$

$$\frac{y(x-2)}{x-2} = \frac{4x+5}{x-2}$$

$$y = \frac{4x+5}{x-2}$$

Step 4

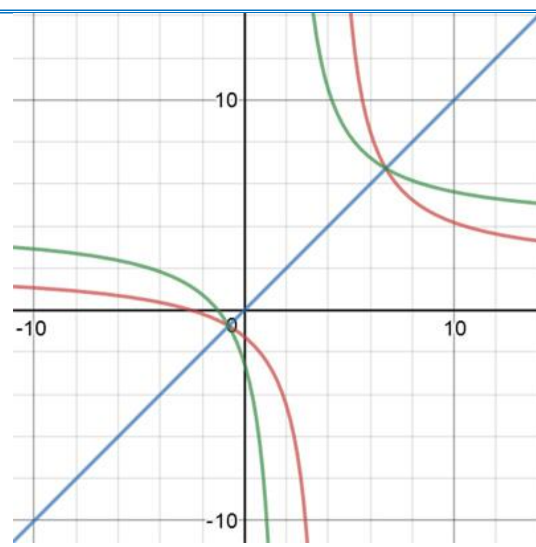
Replace y with $f^{-1}(x)$.

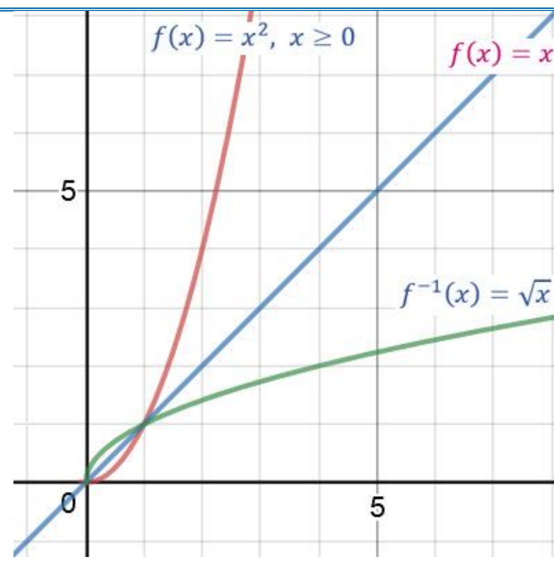
$$f^{-1}(x) = \frac{4x+5}{x-2}$$

So, the inverse of $f(x) = \frac{2x+5}{x-4}$ is

$$f^{-1}(x) = \frac{4x+5}{x-2}$$

The graph of an inverse function is symmetric to the given function with respect to the line $f(x) = x$.



How to Find the Inverse of a Function	
Example 3 Find the inverse of the function $f(x) = x^2,\ x \geq 0$	
Step 1 Replace $f(x)$ with y. $f(x) = x^2$ $y = x^2$	Step 2 Interchange x and y. $x = y^2$
Step 3 Solve for y. $(x)^{\frac{1}{2}} = (y^2)^{\frac{1}{2}}$ $\sqrt{x} = y$	Step 4 Replace y with $f^{-1}(x)$. $y = x$ $f^{-1}(x) = \sqrt{x}$ So, the inverse of $f(x) = x^2,\ x \geq 0$ is $f^{-1}(x) = \sqrt{x}$
The graph of an inverse function is symmetric to the given function with respect to the line $f(x) = x$.	

Not all the functions have an inverse function.

A function has an inverse, if there is no horizontal line that crosses the function's graph in more than one point.